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Electromagnetic waves in linear media

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Abstract. Using the Laplace transform and the Beltrami–Moses fields we present a method for analysing the propagation of electromagnetic waves with arbitrary time dependence in a medium with constant permittivity and permeability. The initial value problem for the TE, TM and TEM solutions of Maxwell's equations is discussed for different point sources in order to obtain the solutions for arbitrary initial data by the Green function method.

1. Introduction

The steady-state electromagnetism used successfully by Müller [1] has two physical drawbacks, stressed recently by Harmuth [2], since it lies outside the conservation law of energy and outside the causality law. This uncomfortable situation suggests that we look for solutions of Maxwell's equations in the time domain rather than in the frequency domain, especially since digital technology for electromagnetism is becoming available.

This problem has previously been discussed by authors such as Felsen [3], Harmuth [2, 4], and recently Weston [5] and Kristensson and Rikte [6] using different techniques. We will consider electromagnetic waves with arbitrary time dependence and propagating in a medium with constant permittivity and permeability, ε and μ [7].

The particular form of the Maxwell–Heaviside equations makes it possible to develop the electromagnetic field on the Beltrami eigenvectors of the curl operator and to use the 1D Laplace transform with respect to time. In addition, we use the 3D complex formalism of electromagnetism [8] developed elsewhere in a different context; here it appears only as a tool facilitating analysis [9].

The mathematical formulation of the problem to be discussed is as follows. Let $\tilde{\Lambda}(x, t)$ be the 3D complex vector [8]

$$\tilde{\Lambda}(x,t) = \varepsilon^{1/2} E(x,t) + \mathrm{i}\mu^{1/2} H(x,t) \tag{1}$$

where E, H are the electric and magnetic fields, $\mathrm{i} = (-1)^{1/2}$ and ε and μ are the permittivity and permeability, respectively (some other definitions of $\tilde{\Lambda}$ are possible [9]). Then, the Maxwell–Heaviside equations

$$\begin{aligned} \nabla \wedge E(x,t) &= -\mu c^{-1}\partial_t H(x,t) \qquad \nabla\cdot H(x,t) = 0 \\ \nabla \wedge H(x,t) &= \varepsilon c^{-1}\partial_t E(x,t) \qquad \nabla\cdot E(x,t) = 0 \end{aligned} \tag{2}$$

become

$$\nabla \wedge \tilde{\Lambda}(x,t) = \mathrm{i}nc^{-1}\partial_t \tilde{\Lambda}(x,t) \qquad \nabla\cdot\tilde{\Lambda}(x,t) = 0 \tag{3a}$$

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where ∇ is the nabla symbol, where $n = (\varepsilon\mu)^{1/2}$ is the refractive index and c is the velocity of light. Then, one looks for the solutions of equation (3a) satisfying the initial condition

$$\tilde{\Lambda}(x, t)_{t=0} = \Lambda(x, 0). \tag{3b}$$

Now applying the Laplace transform [10]

$$F(s) = \int_0^{\infty} e^{-st} \tilde{F}(t)\, dt \qquad \mathrm{Re}\, s > 0 \tag{4}$$

to equation (3) gives

$$\nabla \wedge \Lambda(x, s) - inc^{-1}[s\Lambda(x, s) - \Lambda(x, 0)] = 0 \qquad \nabla \cdot \Lambda(x, s) = 0 \tag{5}$$

so that we now need to solve equation (5). This is achieved with the help of the Beltrami–Moses fields to be discussed in the next section and where the time-dependent solutions are the inverse Laplace transform of the s-dependent solutions.

2. Beltrami–Moses fields

The Beltrami vectors $B(x)$ are the eigenvectors of the curl operator

$$\nabla \wedge B(x) = kB(x) \qquad x \in \mathbb{R}^3 \tag{6}$$

with the eigenvalue k. The Beltrami fields are used in many branches of physics (mainly in fluid mechanics); they were introduced by Moses [11] into electromagnetism in a particular form which justifies our calling them Beltrami–Moses fields. Moses has proved that in his formulation the spectrum of the curl operator comprises the three points $k = 0, \pm 1$. We give a succinct account of the properties of the Beltrami–Moses fields using a recent paper by Moses and Prosser [12].

The Beltrami–Moses fields are 3D complex vectors $\phi(x, p; k)$ depending on x and on a vector p in momentum space such that

$$\nabla \wedge \phi(x, p; k) = k|p|\phi(x, p; k) \qquad x = (x_1, x_2, x_3) \qquad p = (p_1, p_2, p_3) \tag{7}$$

with the eigenvalues $k = 0, \pm 1$ and $|p|$ the modulus of the vector p.

The complex vectors ϕ satisfy the orthogonality and completeness relations ($j, m = 1, 2, 3$)

$$\int dx\, \bar{\phi}(x, p; k) \cdot \phi(x, p'; l) = \delta(p - p')\delta_{kl} \tag{8a}$$

$$\sum_k \int dp\, \bar{\phi}_j(x, p; k)\phi_m(x', p; k) = \delta(x - x')\delta_{jm}. \tag{8b}$$

In these relations the bar and the dot denote respectively the complex conjugation and the scalar product, $\delta(\,)$ is the Dirac distribution, $\delta_{..}$ the Kronecker symbol and ϕ_j, $\bar{\phi}_m$ are the components of the vectors ϕ and $\bar{\phi}$.

One has the relations

$$\nabla \cdot \phi(x, p; k) = -\mathrm{i}(2\pi)^{-3/2} \mathrm{e}^{\mathrm{i}p\cdot x} \delta_{k0} \tag{9a}$$

$$\bar{\phi}(x, p; k) = -\mathrm{e}^{-2\mathrm{i}kv} \phi(x, p; k) \tag{9b}$$

where v is the polar angle of p and

$$p = |p|(\sin u \cos v, \sin u \sin v, \cos u). \tag{10}$$

With $\hat{p} = |p|p^{-1}$ we have explicitly

$$\phi(x, p; 0) = -(2\pi)^{-3/2} \mathrm{e}^{\mathrm{i}p\cdot x} \hat{p} \tag{11a}$$

$$\phi_j(x, p; \pm 1) = -(2\pi)^{-3/2} \mathrm{e}^{\mathrm{i}p\cdot x} \psi_j(\hat{p}, \pm 1) \qquad j = 1, 2, 3. \tag{11b}$$

and

$$\begin{aligned} \psi_1(\hat{p}, k) &= -k + (kp_1 + \mathrm{i}p_2)p_1(1 + p_3)^{-1} \\ \psi_2(\hat{p}, k) &= -\mathrm{i} + (kp_1 + \mathrm{i}p_2)p_2(1 + p_3)^{-1} \\ \psi_3(\hat{p}, k) &= kp_1 + \mathrm{i}p_2. \end{aligned} \tag{12}$$

Any well behaved 3D complex vector $V(x)$ has the representation

$$V(x) = \sum_{k=0,\pm 1} \int \phi(x, p; k) f(p, k)\, \mathrm{d}p \tag{13}$$

where $f(p, k)$ is a scalar function. Because of the completeness and orthogonality relations (8) the expansion coefficient can be obtained from

$$f(p, k) = \int \bar{\phi}(x, p; k) \cdot V(x)\, \mathrm{d}x. \tag{14}$$

We recently used the Beltrami–Moses fields to solve the so-called Harmuth problem [13]; a more general discussion of these fields may be found in that paper.

3. Initial value problems in electromagnetism

The solutions of initial value problems for Maxwell's equations are the bounded solutions to equation (5) with specified data $\tilde{\Lambda}(x, 0)$. Let us apply the expansion (13) to the complex vector fields $\Lambda(x, s)$ and $\tilde{\Lambda}(x, 0)$:

$$\Lambda(x, s) = \sum_k \int \phi(x, p; k) f(p, s; k)\, \mathrm{d}p \tag{15a}$$

$$\tilde{\Lambda}(x, 0) = \sum_k \int \phi(x, p; k) \tilde{f}(p, 0; k)\, \mathrm{d}p. \tag{15b}$$

Substituting (15) into (5) and taking (9*a*) into account we obtain the algebraic system of equations

$$k|p| f(p, s; k) - \mathrm{i}nc^{-1}(sf(p, s; k) - \tilde{f}(p, 0; k)) = 0 \tag{16a}$$

$$f(p, s; 0) = 0. \tag{16b}$$

From (16*a*) and (16*b*) we obtain

$$f(p,0;0)=0 \tag{17a}$$
$$f(p,s;k=\pm 1)=\tilde{f}(p,0;k)(s+\mathrm{i}cn^{-1}|p|)^{-1}. \tag{17b}$$

Substituting (17*b*) and (15) gives the solutions of Maxwell's equations in the s-domain:

$$\Lambda(x,s)=\sum_k \int \phi(x,p;k)f(\tilde{p},0;k)(s+\mathrm{i}kc|p|n^{-1})^{-1}\,\mathrm{d}p \tag{18}$$

where, according to (14), $\tilde{f}(p,0;k)$ is obtained from the initial data $\Lambda(x,0)$ for the electromagnetic field by the relation

$$\tilde{f}(p,0;k)=\int \phi(x,p;k)\cdot\Lambda(x,0)\,\mathrm{d}x. \tag{19a}$$

Using (11*a*) and according to (19) the condition (17) becomes

$$\int \mathrm{e}^{\mathrm{i}p\cdot x}p\cdot\Lambda(x,0)\,\mathrm{d}x=0. \tag{19b}$$

To summarize, provided that the initial data satisfy the condition (19*b*) the solution of the problem in the s-domain is given by (18). Consequently the time-dependent solution is the inverse Laplace transform $L^{-1}[\Lambda(x,s)]$ of (18), that is [10]

$$L^{-1}\{\Lambda(x,s)\}=\Lambda(x,t)=(2\pi\mathrm{i})^{-1}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty}\mathrm{e}^{st}\Lambda(x,s)\,\mathrm{d}s \tag{20}$$

where d is an abscissa in the complex s-plane with all the singularities of $\Lambda(x,s)$ on its left-hand side. In most cases one has to compute numerically the inverse Laplace transform (20). This task is often made difficult since the inversion is, in Hadamard's terminology [14], an 'ill posed' problem. It would be interesting to obtain the solutions of Maxwell's equations for point sources in order to obtain the solutions for arbitrary initial data by the Green's function method. We discuss these particular solutions in the next section.

Remark. Let us also note the Widder inverse formula [15] which involves only the values of $F(s)$ at the real axis of s:

$$\tilde{F}(t)=\lim_{n\to\infty}\left\{\frac{s^{n+1}}{n!}\left(\frac{-\mathrm{d}}{\mathrm{d}s}\right)^n\frac{F(s)}{s}\right\}_{s=n/t}\qquad t>0 \tag{21}$$

which we may also write as

$$\tilde{F}(t)=\lim_{n\to\infty}\left\{\frac{\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{F(s)}{s}}{\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{1}{s}}\right\}_{s=n/t}. \tag{22}$$

The Widder formula gives the interesting relation

$$\int_{t-0}^{t+0}\tilde{F}(\tau)\,\mathrm{d}\tau=\lim_{n\to\infty}(-\mathrm{e}/t)^n\left\{\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{F(s)}{s}\right\}_{s=n/t}\qquad t>0 \tag{23}$$

which determines values of t for which $f(t)$ is impulsive; elsewhere the limit is automatically zero.

4. Initial value problems with point sources

4.1. TEM pulses

We first assume that the scalar functions $\tilde{f}(p, 0; k)$, $k = \pm 1$, is

$$\tilde{f}(p, 0; k) = [a(\xi)k + b(\xi)] \exp(-\mathrm{i}p_3\xi)\delta(p_1)\delta(p_2) \tag{24}$$

where a and b are arbitrary complex functions and $\delta(p_{1,2})$ is the Dirac distribution. Substituting (24) into (15*b*) and using (11*b*) and (12) a simple calculation gives

$$\begin{aligned} \tilde{\Lambda}_1(x, 0) &= \sqrt{2}\, a(\xi)\delta(z - \xi) \\ \tilde{\Lambda}_2(x, 0) &= \mathrm{i}\sqrt{2}\, b(\xi)\delta(z - \xi) \\ \tilde{\Lambda}_3(x, 0) &= 0. \end{aligned} \tag{25a}$$

Since $p_1 = p_2 = 0$ implies $|p| = |p_3|$ and $\psi_1 = -k$, $\psi_2 = -\mathrm{i}$ and $\psi_3 = 0$, one checks easily that the condition (19*b*) is fulfilled since

$$\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}p\cdot x}[p_1 a(\xi) + \mathrm{i}p_2 b(\xi)]\delta(z - \xi) = 0. \tag{25b}$$

Now substituting (24) into (18) we obtain

$$\begin{aligned} \Lambda_1(x, s) &= \mathrm{i}(4\pi)^{-1/2}\sum_k \int \mathrm{d}p_3 \{a(\xi) + kb(\xi)\}\{s + \mathrm{i}kcp_3n^{-1}\}^{-1} \exp[\mathrm{i}p_3(z - \xi)] \\ \Lambda_2(x, s) &= \mathrm{i}(4\pi)^{-1/2}\sum_k \int \mathrm{d}p_3 \{b(\xi) + ka(\xi)\}\{s + \mathrm{i}kcp_3n^{-1}\}^{-1} \exp[\mathrm{i}p_3(z - \xi)] \\ \Lambda_3(x, s) &= 0. \end{aligned} \tag{26}$$

Then, using the Fourier transform relation [16]

$$(2\pi)^{-1/2}\int_{-\infty}^{+\infty} \mathrm{d}p\, \mathrm{e}^{\mathrm{i}p\cdot x}(\alpha \pm \mathrm{i}p)^{-1} = \mathrm{e}^{\pm\alpha x}U(x) \qquad \mathrm{Re}\,\alpha > 0 \tag{27}$$

where U is the step function, we obtain from (26) the following:

$$\begin{aligned} \Lambda_1(x, s) &= nc^{-1}2a(\xi)\exp(-nsz - \xi/c) \\ \Lambda_2(x, s) &= \mathrm{i}nc^{-1}2b(\xi)\exp(-nsz - \xi/c) \\ \Lambda_3(x, s) &= 0 \end{aligned} \tag{28}$$

and since we have [10]

$$L^{-1}(\mathrm{e}^{nsz/c}) = \delta(t + nz/c) \tag{29}$$

the inverse Laplace transform of (28) is

$$\begin{aligned} \tilde{\Lambda}_1(x, t) &= nc^{-1}\sqrt{2}\,\delta[t + n(z - \xi)/c] \\ \tilde{\Lambda}_2(x, t) &= \mathrm{i}nc^{-1}\sqrt{2}\,\delta[t + n(z - \xi)/c] \\ \tilde{\Lambda}_3(x, t) &= 0. \end{aligned} \tag{30a}$$

The solution (30) represent a TEM electromagnetic pulse propagating in the z-direction. Using the relation

$$\delta[n(z - \xi)/c] = cn^{-1}\delta(z - \xi) \tag{30b}$$

one checks easily that the solution (30) satisfies the initial condition (25).

4.2. Pulsed infinitely long magnetic and electric sources

In these cases one has for $k = \pm 1$

$$\tilde{f}(p, 0; k) = [a(\eta, \xi) + b(\eta, \xi)p] \exp[-\mathrm{i}(p_2\eta + p_3\xi)]\delta(p_1) \tag{31}$$

where a and b are now arbitrary functions of η and ξ. Substituting (31) into (15b) and using (11b) and (12) we obtain

$$\begin{aligned}\tilde{\Lambda}_1(\boldsymbol{x}, 0) &= \sqrt{2}\, a(\eta, \xi)\delta(y-\eta)\delta(z-\xi)\\ \tilde{\Lambda}_2(\boldsymbol{x}, 0) &= \mathrm{i}\sqrt{2}\, b(\eta, \xi)\delta(y-\eta)\delta'(z-\xi)\\ \tilde{\Lambda}_3(\boldsymbol{x}, 0) &= -\mathrm{i}\sqrt{2}\, b(\eta, \xi)\delta'(y-\eta)\delta(z-\xi)\end{aligned} \tag{32}$$

where δ' is the derivative of the Dirac distribution since

$$|p| = (p_2^2 + p_3^2)^{1/2} \qquad \text{and} \qquad \psi_1 = -k,\ \psi_2 = -\mathrm{i}p_3,\ \psi_3 = \mathrm{i}p_2.$$

Then, using the relation

$$\int_{-\infty}^{+\infty} f(y)\delta'(x-y)\,\mathrm{d}y = -f'(x) \tag{33}$$

one proves easily that the condition (19b) is fulfilled since

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{\mathrm{i}p\cdot\boldsymbol{x}}\{p_1 a(\eta, \xi)\delta(y-\eta)\delta(z-\xi) + \mathrm{i}p_2 b(\eta, \xi)\delta(y-\eta)\delta'(z-\xi)\\ - \mathrm{i}p_3 b(\eta, \xi)\delta'(y-\eta)\delta(z-\xi)\} = 0.\end{aligned}$$

Now, substituting (31) into (18) gives

$$\begin{aligned}\Lambda_1(\boldsymbol{x}, s) = (8\pi^2)^{-1/2}\sum_k \int \mathrm{d}p_2\,\mathrm{d}p_3 \exp\{\mathrm{i}p_3(z-\xi)\}\exp\{\mathrm{i}p_2(y-\eta)\}\\ \times (a + kb|p|)(s + \mathrm{i}kcn|p|)^{-1}\end{aligned} \tag{34}$$

and similar expressions for $\Lambda_2(\boldsymbol{x}, s)$ with $\mathrm{d}p_3$ replaced by $\mathrm{i}p_3\,\mathrm{d}p_3$, and for $\Lambda_3(\boldsymbol{x}, s)$ with $\mathrm{d}p_2$ replaced by $-\mathrm{i}p_2\,\mathrm{d}p_2$. In these expressions $|p| = (p_2^2 + p_3^2)^{1/2}$. Then, using the variables

$$z - \xi = r\cos\phi \qquad y - \eta = r\sin\phi \qquad p_3 = |p|\cos\theta \qquad p_2 = |p|\sin\theta \tag{35a}$$

and writing

$$\Lambda(\boldsymbol{x}, s) = \Lambda(\boldsymbol{x}, s; a) + \Lambda(\boldsymbol{x}, s; b) \tag{35b}$$

$$I(|p|) = \int_0^{2\pi} \mathrm{d}\theta \exp\{\mathrm{i}r|p|\cos(\theta - \phi)\} \tag{35c}$$

we obtain

$$\Lambda_1(\boldsymbol{x}, s; a) = a(2\pi^2)^{-1/2} s \int \mathrm{d}|\boldsymbol{p}|\, |\boldsymbol{p}| I(|\boldsymbol{p}|)(s^2 + c^2 n^{-2}|\boldsymbol{p}|^2)^{-1}$$

$$\Lambda_2(\boldsymbol{x}, s; a) = a n^{-1} c (2\pi^2)^{-1/2} \int \mathrm{d}|\boldsymbol{p}|\, |\boldsymbol{p}|^2 I(|\boldsymbol{p}|)(s^2 + c^2 n^{-2}|\boldsymbol{p}|^2)^{-1} \qquad (36a)$$

$$\Lambda_3(\boldsymbol{x}, s; a) = -a n^{-1} c (2\pi^2)^{-1/2} \int \mathrm{d}|\boldsymbol{p}|\, |\boldsymbol{p}|^2 I(|\boldsymbol{p}|)(s^2 + c^2 n^{-2}|\boldsymbol{p}|^2)^{-1}$$

and

$$\Lambda_1(\boldsymbol{x}, s; b) = -\mathrm{i} n^{-1} c (2\pi)^{-1/2} \int \mathrm{d}|\boldsymbol{p}|\, |\boldsymbol{p}|^2 I(|\boldsymbol{p}|)(s^2 + c^2 n^{-2}|\boldsymbol{p}|^2)^{-1}$$

$$\Lambda_2(\boldsymbol{x}, s; b) = \mathrm{i} s (2\pi)^{-1/2} \int \mathrm{d}|\boldsymbol{p}|\, |\boldsymbol{p}|^2 I(|\boldsymbol{p}|)(s^2 + c^2 n^{-2}|\boldsymbol{p}|^2)^{-1} \qquad (36b)$$

$$\Lambda_3(\boldsymbol{x}, s; b) = -\mathrm{i} s (2\pi)^{-1/2} \int \mathrm{d}|\boldsymbol{p}|\, |\boldsymbol{p}|^2 I(|\boldsymbol{p}|)(s^2 + c^2 n^{-2}|\boldsymbol{p}|^2)^{-1}.$$

From (36*a*) and (36*b*) we obtain the relations

$$\begin{aligned} \cos\phi \Lambda_2(\boldsymbol{x}, s; a) - \sin\phi \Lambda_3(\boldsymbol{x}, s; a) &= -\mathrm{i}c/sn \partial_r \Lambda_1(\boldsymbol{x}, s; a) \\ \sin\phi \Lambda_2(\xi, s; a) + \cos\phi \Lambda_3(\boldsymbol{x}, s; a) &= -\mathrm{i}c/snr \partial_\phi \Lambda_1(\boldsymbol{x}, s; a) \end{aligned} \qquad (37a)$$

and

$$\begin{aligned} r^{-1}\partial_\phi \Lambda_3(\boldsymbol{x}, s; b) - \partial_r \Lambda_2(\boldsymbol{x}, s; b) &= -\mathrm{i}ns/c \cos\phi \Lambda_1(\boldsymbol{x}, s; b) \\ r^{-1}\partial_\phi \Lambda_2(\boldsymbol{x}, s; b) + \partial_r \Lambda_3(\boldsymbol{x}, s; b) &= \mathrm{i}ns/c \sin\phi \Lambda_1(\boldsymbol{x}, s; b) \end{aligned} \qquad (37b)$$

so that one has just to compute $\Lambda_1(\boldsymbol{x}, s; a)$ and $\Lambda_2(\boldsymbol{x}, s; b)$.

In the appendix we prove the following results.

(i) The expressions of $\Lambda_1(\boldsymbol{x}, s; a)$ and $\Lambda_2(\boldsymbol{x}, s; b)$ are

$$\Lambda_1(\boldsymbol{x}, s; a) = 2^{1/2} a n^2 c^{-2} s K_0(nrs/c) \qquad (38a)$$

$$\Lambda_2(\boldsymbol{x}, s; b) = -2^{1/2} b n^2 c^{-2} s \mathrm{e}^{-\alpha}(1 - \mathrm{e}^{-\alpha/2})^{-1} Q^1_{1/2}(\cosh\alpha) - \mathrm{i}2^{1/2} bn/cr (1 - \mathrm{e}^{-2\alpha})^{-1} \qquad (38b)$$

where K_0 is the modified Bessel function of the second kind of order zero, $Q^1_{-1/2}$ is an associated Legendre function of the second kind, and $\mathrm{e}^\alpha = 2c/nrs$.

(ii) The inverse Laplace transforms of (38*a*) and (38*b*) are

$$\tilde{\Lambda}_1(\boldsymbol{x}, t; a) = 2^{1/2} n^2 c^{-2} \partial_t \{U(t - nr/c)(t^2 - n^2 c^{-2} r^2)^{-1/2}\} \qquad (39a)$$

$$\tilde{\Lambda}_2(\boldsymbol{x}, t; b) = -2^{1/2} n^2 c^{-2} \tilde{E}(t - nr/c) - \mathrm{i}2^{3/2} r^{-2} \sinh(2ct/nr) \qquad (39b)$$

where U is the step function and $\tilde{E}$ is the inverse Laplace transform

$$\tilde{E} = L^{-1}\{s \mathrm{e}^{\alpha/2}(1 - \mathrm{e}^{-2\alpha})^{-1} Q^1_{-1/2}(\cosh a)\}.$$

In (38) and (39) a and b are written for $a(\eta, \xi)$ and $b(\eta, \xi)$ respectively. The function $\tilde{E}$ has no simple analytical expression.

(iii) The initial condition (32) for $\tilde{\Lambda}(\boldsymbol{x}, t) = \tilde{\Lambda}(\boldsymbol{x}, t; a) + \tilde{\Lambda}(\boldsymbol{x}, t; b)$ is fulfilled.

To complete the solution of this particular initial value problem one has still to compute the components $\tilde{\Lambda}_2(x,t)$ and $\tilde{\Lambda}_3(x,t)$. Now, since $\Lambda_1(x,s;a)$ does not depend upon ϕ, and since $s\partial_s F(nrs/c) = r\partial_r F(nrs/c)$ we obtain from (37*a*)

$$\Lambda_2(x,s;a) = -\mathrm{i}c\cos\phi/sn\partial_r\Lambda_1(x,s;a) = -\mathrm{i}c\cos\phi/rn\partial_s\Lambda_1(x,s;a)$$
$$\Lambda_3(x,s;a) = \mathrm{i}c\sin\phi/sn\partial_r\Lambda_1(x,s;a) = \mathrm{i}c\sin\phi/rn\partial_s\Lambda_1(x,s;a)$$

and using the relation [10] $\mathrm{d}/\mathrm{d}sF(s) = L(-tF(t))$ we find

$$\begin{aligned}\tilde{\Lambda}_2(x,t;a) &= \mathrm{i}\cos\phi ct/nr\tilde{\Lambda}_1(x,t;a)\\ \tilde{\Lambda}_3(x,t;a) &= -\mathrm{i}\sin\phi ct/nr\tilde{\Lambda}_1(x,t;a).\end{aligned} \tag{40}$$

Substituting (39*a*) into (40) gives the explicit expression of the components $\tilde{\Lambda}_2(x,t;a)$ and $\tilde{\Lambda}_3(x,t;a)$. Now, the solution of (37*b*) is

$$\begin{aligned}\Lambda_3(x,s;b) &= \sin\phi A(x,s;b)\\ \Lambda_2(x,s;b) &= \cos\phi A(x,s;b)\end{aligned} \tag{41a}$$

where the function $A(x,s;b)$ satisfies the equation

$$r^{-1}\partial_\phi A(x,s;b) - \partial_r A(x,s;b) = -inc^{-1}s\Lambda_1(x,s;b)$$

which reduces to

$$\partial_r A(x;s;b) = inc^{-1}s\Lambda_1(x,s;b) \tag{41b}$$

since $\Lambda_1(x,s;b)$ is a function of nrs/c that does not depend on ϕ, so we obtain from (41*b*)

$$\begin{aligned}A(x,s;b) &= inc^{-1}\int\Lambda_1(x,s;b)s\,\mathrm{d}r\\ &= inrc^{-1}\int\Lambda_1(x,s;b)\,\mathrm{d}s\end{aligned} \tag{42}$$

because $s\,\mathrm{d}r = r\,\mathrm{d}s$. Then, using the relation [10] $L^{-1}\int F(s)\,\mathrm{d}s = F(t)t^{-1}$ we obtain $\tilde{A}(x,t;b) = inr(ct)^{-1}\tilde{\Lambda}_1(x,t;b)$, and according to (41)

$$\begin{aligned}\tilde{\Lambda}_2(x,t;b) &= -\mathrm{i}\cos\phi nr/ct\tilde{\Lambda}_1(x;t;b)\\ \tilde{\Lambda}_3(x,t;b) &= \mathrm{i}\sin\phi nr/ct\tilde{\Lambda}_1(x,t;b).\end{aligned} \tag{43}$$

Substituting (39*b*) into (43) gives the explicit expressions for $\tilde{\Lambda}_2(x,t;b)$ and $\tilde{\Lambda}_3(x,t;b)$. The real and imaginary parts of the expressions (39), (40) and (41) represent respectively TE and TM electromagnetic pulses with the transverse component orthogonal to the yz-plane and satisfying the initial condition (32).

We may also generate initial data in a plane. Using the polar coordinates

$$y = r\cos\theta \qquad z = r\sin\theta \qquad p_2 = |p|\cos\phi \qquad p_3 = |p|\sin\phi \tag{44}$$

and the initial scalar function

$$\tilde{f}(p,0,k) = (ak+b)\delta(p_1)|p|^{-1}\delta(|p|-k)\delta(\phi-\phi_0) \tag{45}$$

where a and b are arbitrary constants. We obtain easily from (15)

$$\begin{aligned}\tilde{\Lambda}_1(\boldsymbol{x},0) &= a\exp\{ikr\cos(\phi-\phi_0)\}\\ \tilde{\Lambda}_2(\boldsymbol{x},0) &= b\cos\phi_0\exp\{ikr\cos(\phi-\phi_0)\}\\ \tilde{\Lambda}_3(\boldsymbol{x},0) &= -b\sin\phi_0\exp\{ikr\cos(\phi-\phi_0)\}\end{aligned} \tag{46}$$

which represents a sinusoidal source in the xy-plane. Another simple example is obtained with

$$\tilde{f}(\boldsymbol{p},0;k) = (ak+b)(p_2p_3)^{-1}\delta(p_1)[1-\exp(-ip_2\eta)][1-\exp(-ip_3\xi)]. \tag{47}$$

Substituting (47) into (15) we obtain

$$\begin{aligned}\tilde{\Lambda}_1(\boldsymbol{x},0) &= -a[U(y)-U(y-\eta)][U(z)-U(z-\xi)]\\ \tilde{\Lambda}_2(\boldsymbol{x},0) &= b[U(y)-U(y-\eta)][\delta(z)-\delta(z-\xi)]\\ \tilde{\Lambda}_3(\boldsymbol{x},0) &= b[\delta(y)-\delta(y-\eta)][U(z)-U(z-\xi)]\end{aligned} \tag{48}$$

corresponding to initial data on the rectangle with vertices at (0,0), $(\eta,0)$, $(\eta\,\xi)$ and $(0,\xi)$.

5. TEM, TE and TM pulses

To obtain more general TEM impulse solutions we consider the function

$$\tilde{f}(\boldsymbol{p},0;k) = \delta(p_1)\delta(p_2)\int_{-\infty}^{+\infty} d\xi\,[a(\xi)k+b(\xi)]\exp(-ip_3\xi) \tag{49}$$

instead of (24) and we assume that

(i) the functions $a(\xi)$ and $b(\xi)$ are continuous,
(ii) for $k=\pm 1$ the integral in (49) exists and converges absolutely.

We can then exchange the order of integration on $\boldsymbol{p}$ and ξ in (15b) and (18) so that we obtain from (25) and (28)

$$\tilde{\Lambda}_1(\boldsymbol{x},0)=\sqrt{2}\,a(z) \qquad \tilde{\Lambda}_2(\boldsymbol{x},0)=i\sqrt{2}\,b(z) \qquad \tilde{\Lambda}_3(\boldsymbol{x},0)=0 \tag{50}$$

and

$$\begin{aligned}\Lambda_1(\boldsymbol{x},s) &= \sqrt{2}\,n/c\int a(\xi)\exp(-nsz-\xi/c)\,d\xi\\ \Lambda_2(\boldsymbol{x},s) &= i\sqrt{2}\,n/c\int b(\xi)\exp(-nsz-\xi/c)\,d\xi\\ \Lambda_3(\boldsymbol{x},s) &= 0\end{aligned} \tag{51}$$

which is the solution in the s-plane of the initial value problem with initial data (50). From the definition of the Laplace transform we obtain

$$\Lambda_1(\boldsymbol{x},s) = \sqrt{2}\,nc^{-1}\int_{-\infty}^{+\infty} d\xi\,a(\xi)\int_0^{\infty} dt\,e^{-st}\delta[t+n(z-\xi)/c] \tag{52}$$

which we can write (given the absolute convergence of the first integral) as

$$\Lambda_1(\boldsymbol{x}, s) = \sqrt{2}nc^{-1}\int \mathrm{d}t\mathrm{e}^{-st}\int \mathrm{d}\xi a(\xi)\delta[t + nc^{-1}(z - \xi)]$$
$$= 2\int_0^\infty \mathrm{d}t\mathrm{e}^{-st}a(c + ct/n). \tag{53}$$

There is a similar expression for $\Lambda_2(\boldsymbol{x}, s)$ so that the inverse Laplace transform of (51) is

$$\tilde{\Lambda}_1(\boldsymbol{x}, t) = \sqrt{2}a(ct/n + z)$$
$$\tilde{\Lambda}_2(\boldsymbol{x}, t) = \mathrm{i}\sqrt{2}b(ct/n + z) \tag{54}$$
$$\tilde{\Lambda}_3(\boldsymbol{x}, t) = 0.$$

The expressions (54) give, for $t > 0$, the most general form of a TEM electromagnetic pulse with the initial conditions (50) and propagation in the z-direction.

For the TE and TM electromagnetic pulses we use the function

$$\tilde{f}(\boldsymbol{p}, 0; k) = \delta(p_1)\int\int_{-\infty}^{\infty} \mathrm{d}\eta\,\mathrm{d}\xi[a(\eta, \xi) + b(\eta, \xi)]|\boldsymbol{p}|\exp[-\mathrm{i}(p_2\eta + p_3\xi)] \tag{55}$$

instead of (31) and again we assume the continuity of the functions a and b and the absolute convergence of the double integral in (55). The initial conditions for the electromagnetic field are

$$\tilde{\Lambda}_1(\boldsymbol{x}, 0) = \sqrt{2}a(y, z) \qquad \tilde{\Lambda}_2(\boldsymbol{x}, 0) = -\mathrm{i}\sqrt{2}b(y, z) \qquad \tilde{\Lambda}_3(\boldsymbol{x}, 0) = \mathrm{i}\sqrt{2}b(y, z). \tag{56}$$

Since the solutions for the Dirac TE and TM pulses are rather intricate we assume for simplicity that $b(y, z) = 0$. Using (38a), the expression of $\Lambda_1(\boldsymbol{x}, s; a)$ in terms of the Bessel function K_0 is

$$\Lambda_1(\boldsymbol{x}, s; a) = \sqrt{2}n^2c^{-2}s\int\int_{-\infty}^{\infty} \mathrm{d}\eta\,\mathrm{d}\xi a(\eta, \xi)K_0(nrs/c) \tag{57}$$

which becomes, with the variables (35a),

$$\Lambda_1(\boldsymbol{x}, s; a) = \sqrt{2}n^2c^{-2}\int \mathrm{d}\phi\int r\,\mathrm{d}ra(y - r\sin\phi, z - r\cos\phi)sK_0(nrs/c). \tag{58}$$

Now according to the relation (A12) in the appendix we have

$$sK_0(nrs/c) = L\{\partial_t[U(t - nr/c)(t^2 - n^2r^2c^{-2})^{-1/2}]\} \tag{59}$$

and arguing as previously we obtain

$$\tilde{\Lambda}_1(\boldsymbol{x}, t) = 2^{-1/2}n^2c^{-2}\pi^{-1}\int_0^{2\pi} \mathrm{d}\phi\int_0^\infty r\,\mathrm{d}ra(y - r\sin\phi, z - r\cos\phi)$$
$$\times\,\partial_t\{U(t - nr/c)(t^2 - n^2r^2c^{-2})^{-1/2}\}$$
$$= \sqrt{2}n(\pi c)^{-1}\partial_t\int_0^{2\pi} \mathrm{d}\phi\int_0^\infty r\,\mathrm{d}ra(y - r\sin\phi, z - r\cos\phi)(n^{-2}c^2t^2 - r^2)^{-1/2}. \tag{60}$$

Applying (43) to (60) supplies the other two components $\tilde{\Lambda}_2(\boldsymbol{x}, t; a)$ and $\tilde{\Lambda}_3(\boldsymbol{x}, t; a)$. Thus we have obtained the general TE–TM electromagnetic pulse with the initial condition (56) for the case $b = 0$. For $b \neq 0$ calculations are more intricate and would require a numerical integration of the Laplace transform.

6. Discussion

The Laplace transform is a natural tool to tackle electromagnetic initial value problems; in the past the Laplace transform has been used by some authors, but mainly for simple problems [6, 9]. Moses [11] was the first to stress the role of the Beltrami vectors in electromagnetism. His results, largely ignored until the recent blossoming of work on chiral media, are now finding more applications (Lakhtakia [17–20]) for two reasons:

(i) as we have seen the Beltrami vectors are very well suited to the 3D complex formalism of electromagnetism;

(ii) as previously discussed [8] this formalism is a powerful tool in chiral media because it is covariant under the proper Lorentz group excluding space inversion.

It was natural to combine the Laplace transform and Beltrami vectors to discuss the Cauchy problem for Maxwell's equations. The results of the present work may be used with a great variety of sources [13], and they should be of interest to applications such as interferences of digital electromagnetic signals [21]. The solution in the s-domain is generally easy, but in most cases one has to perform the inverse Laplace transform numerically. There now exist powerful codes [22] to do this job, but one must check the results carefully because this inversion is technically an 'ill-posed' problem [14].

Appendix

Let us consider the expression (36) for $\Lambda_1(\boldsymbol{x}, s; a)$:

$$\Lambda_1(\boldsymbol{x}, s; a) = as(2\pi^2)^{-1/2} \int_0^\infty |\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{p}|(s^2 + c^2 n^{-2}|\boldsymbol{p}|)^{-1} \int_0^{2\pi} \mathrm{d}\theta \exp[-\mathrm{i}r|\boldsymbol{p}|\cos(\theta - \phi)]. \tag{A1}$$

Using the well known relation [23]

$$(2\pi)^{-1} \int_0^{2\pi} \mathrm{d}\theta \exp[\mathrm{i}r|\boldsymbol{p}|\cos(\theta - \phi)] = J_0(r|\boldsymbol{p}|) \tag{A2}$$

where J_0 is the Bessel function of the first kind of order zero, we obtain

$$\Lambda_1(\boldsymbol{x}, s; a) = \sqrt{2}\, a n^2 c^{-2} s \int_0^\infty |\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{p}| J_0(r|\boldsymbol{p}|)(|\boldsymbol{p}|^2 + n^2 c^{-2} s^2)^{-1}. \tag{A3}$$

This integral is of the Hankel type, and we have [24]

$$\int_0^\infty |\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{p}| J_0(r|\boldsymbol{p}|)(|\boldsymbol{p}|^2 + n^2 c^{-2} s^2)^{-1} = K_0(nrs/c) \tag{A4}$$

where K_0 is the modified Bessel function of the second kind of order zero. Substituting (A4) into (A3) gives

$$\Lambda_1(\boldsymbol{x}, s; a) = \sqrt{2}\, a n^2 c^{-2} s K_0(nrs/c). \tag{A5}$$

Now taking into account (A2) we rewrite the expression (36*b*) for $\Lambda_1(x, s; b)$:

$$\Lambda_1(x,s;b) = -\mathrm{i}\sqrt{2}\,bnc^{-1}\int_0^\infty |p|^2\,\mathrm{d}|p| J_0(r|p|)(|p|^2 + n^2c^{-2}s^2)^{-1}. \qquad \text{(A6)}$$

This last expression is also an integral of Hankel's type, and one has in terms of the hyper-geometric function ${}_1F_2$

$$\int |p|^2\,\mathrm{d}|p| J_0(r|p|)(|p|^2 + n^2c^{-2}s^2)^{-1} = -\pi ns/2c\,{}_1F_2(\tfrac{3}{2}, \tfrac{3}{2}, 1; n^2r^2s^2/4c^2) + r^{-1}\,{}_1F_2(1, \tfrac{1}{2}, \tfrac{1}{2}; n^2r^2s^2/4c^2). \qquad \text{(A7)}$$

With the variable $\mathrm{e}^{\alpha} = 2c/nrs$ we have [23]

$$\begin{aligned} {}_1F_2(1, \tfrac{1}{2}, \tfrac{1}{2}; \mathrm{e}^{-2\alpha}) &= (1 - \mathrm{e}^{-2\alpha})^{-1} \\ {}_1F_2(\tfrac{3}{2}, \tfrac{3}{2}, 1; \mathrm{e}^{-2\alpha}) &= 2\mathrm{i}/\pi \mathrm{e}^{-\alpha/2}(1 - \mathrm{e}^{-2\alpha})^{-1} Q^1_{-1/2}(\cosh\alpha). \end{aligned} \qquad \text{(A8)}$$

Then, substituting (A7) into (A6) and taking into account (A8), we find

$$\Lambda_1(x,s;b) = -\sqrt{2}\,bn^2c^{-2}s\mathrm{e}^{-\alpha/2}(1 - \mathrm{e}^{-2\alpha})^{-1} Q^1_{-1/2}(\cosh\alpha) - \mathrm{i}\sqrt{2}\,bnc^{-1}r(1 - \mathrm{e}^{-2\alpha})^{-1}. \qquad \text{(A9)}$$

We now have to consider the inverse Laplace transform of (A5) and (A9) using [15]

$$L^{-1}[K_0(snr/c)] = (t^2 - n^2r^2/c^2)^{-1/2}U(t - nr/c) \qquad \text{(A10)}$$

where U is the step function. Using the relation [10]

$$sF(s) = \mathrm{d}/\mathrm{d}t\,F(t) + F(0) \qquad \text{(A11)}$$

we obtain from (A10)

$$\begin{aligned} L^{-1}[sK_0(snc^{-1}r)] &= \partial_t\{U(t - nr/c)(t^2 - n^2c^{-2}r^2)^{-1/2}\} \\ &= -tU(t - nr/c)(t^2 - n^2c^{-2}r^2)^{-3/2} + \delta(t - nr/c)(t^2 - n^2c^{-2}r^2)^{-1/2}. \end{aligned} \qquad \text{(A12)}$$

Taking into account (A5) and (A12) the inverse Laplace transform of $\Lambda_1(x, s; a)$ is

$$\tilde{\Lambda}_1(x,t;a) = 2^{1/2}an^2c^{-2}\partial_t\{[t^2 - n^2 - c^{-2}r^2]^{1/2}U(t - ncr^{-1})\}. \qquad \text{(A13)}$$

We have also [16]

$$L^{-1}[(1 - \mathrm{e}^{-2\alpha})^{-1}] = 2c/nr\sinh(2c/nr) \qquad \text{(A14)}$$

and we note

$$\tilde{E}(t - nr/c) = L^{-1}[s\mathrm{e}^{\alpha/2}(1 - \mathrm{e}^{-2\alpha})Q^1_{-1/2}(\cos\alpha)]. \qquad \text{(A15)}$$

It does not seem that there exists an analytical expression for (A15) so $E(t - nr/c)$ has to be computed numerically.

From (A9), (A14) and (A15) we obtain the inverse Laplace transform of $\Lambda_1(x, s; b)$:

$$\tilde{\Lambda}_1(x, t; b) = -\sqrt{2}bn^2c^{-2}E(t - nr/c) - 2\mathrm{i}\sqrt{2}br^{-2}\sinh(2ct/nbr). \quad \text{(A16)}$$

One has still to prove that the initial condition (32) for $\tilde{\Lambda}_1(x, t) = \tilde{\Lambda}_1(x, t; a) + \tilde{\Lambda}_1(x, t; b)$ is fulfilled. First using the relation [10]

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sF(s) \quad \text{(A17)}$$

we obtain from (A6), (A7) and (A16)

$$\lim_{t\to 0} \tilde{\Lambda}_1(x, t; b) = \lim_{s\to\infty} \{-\pi nc^{-1}s_1F_2(\tfrac{3}{2}, \tfrac{3}{2}, 1;' n^2r^2s^2/4c^2) + r^{-1}sF(1, \tfrac{1}{2}, \tfrac{1}{2}; n^2r^2s^2/4c^2\}. \quad \text{(A18)}$$

But for $s \to \infty$ one has [23]

$${}_1F_2(\tfrac{3}{2}, \tfrac{3}{2}, 1; n^2r^2s^2/4c^2) = \mathrm{O}(s^{-3}) \qquad {}_1F_2(1, \tfrac{1}{2}, \tfrac{1}{2}; n^2r^2s^2/4c^2) = \mathrm{O}(s^{-2}) \quad \text{(A19)}$$

where the symbol $\mathrm{O}(s^{-m})$ means that in the neighborhood of infinity the functions ${}_1F_2$ behave as s^{-m}. So according to (A19) the right-hand side of (A18) is zero at infinity. This means that $\tilde{\Lambda}_1(x, 0; b) = 0$. We now obtain from (A13)

$$\lim_{t\to 0} \tilde{\Lambda}_1(x, t; a) = \sqrt{2}a\delta(r)/r = \sqrt{2}a\delta(y - \eta)\delta(z - \xi) \quad \text{(A20)}$$

since one has [15]

$$\delta(r)/r = \delta(y - \eta)\delta(z - \xi) \quad \text{(A21)}$$

and (A21) is exactly the initial condition (32).

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